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Noncommutative quantum electrodynamics in path integral framework

S Bourouaine and A Benslama

Département de Physique, Faculté des Sciences, Université Mentouri, Constantine, Algeria

E-mail: soufi-net@caramail.com and abenslama@wissal.dz

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Abstract
In this paper, the dynamics of a relativistic particle of spin 1/2, interacting with an external electromagnetic field in noncommutative space, is studied in the path integral framework. By adopting the Fradkin–Gitman formulation, the exact Green's function in noncommutative space (NCGF) for the quadratic case of a constant electromagnetic field is computed, and it is shown that its form is similar to its counterpart given in commutative space. In addition, it is deduced that the effect of noncommutativity has the same effect as an additional constant field depending on a noncommutative θ matrix.

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1. Introduction

The noncommutative spacetime is characterized by operators $\hat{x}_\mu$ satisfying the relation

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} = \frac{\mathrm{i}}{\Lambda_{\mathrm{NC}}^2} C^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is the antisymmetric constant matrix which has the dimension of an area. $C^{\mu\nu}$ are dimensionless parameters which are presumably of order unity and Λ_{NC} is the scale energy where the noncommutative effects of the spacetime become relevant. Recently, the appearance of such theories as certain limits of string theory, D-brane and M-theory has motived the interest in studying noncommutative quantum field theories (NCQFT) [1, 2] as well as noncommutative quantum mechanics (NCQM) [3, 4, 7].

There are two different settings in the construction of quantum theories on noncommutative space. The first one is based on the so-called Moyal or star product, and the majority of its applications have been considered in noncommutative quantum electrodynamics (NCQED) [8, 9] and NCQM. The other setting uses the Seiberg–Witten map into non-Abelian gauge theory [14].

The Moyal or star (∗) product of two arbitrary differentiable fields (f and g) is defined as

$$f(x) * g(x) = \left[\exp\left(\frac{\mathrm{i}}{2} \theta^{\mu\nu} \partial_\mu^{(\xi)} \partial_\nu^{(\eta)} f(x+\xi) g(x+\eta) \right) \right]_{\xi=\eta=0}. \tag{2}$$

The deviation from standard gauge theory to noncommutative one causes the violation of Lorentz invariance when θ is considered as a constant matrix, except if this matrix is promoted to a tensor related to the contracted Snyder's Lie algebra [10, 11]. Also, it has been shown that the problem of unitarity appears in the study of quantum theories with time-space noncommutativities ($\theta^{0\mathrm{i}} \neq 0$) [12, 13].

In this work, we investigate the effect of noncommutative space on a Dirac particle interacting with an external electromagnetic field by using the path integral approach. In fact, it is well known that the propagator of a Dirac particle is distinguished from that of the scalar particle by a complicated spin structure. By using Grassmann variables, the description of the Dirac propagator becomes possible to have a path integral representation. This idea was exploited in several works, among which we quote the formalism of Fradkin–Gitman [15, 16], in which the authors have described the evolution of the spinorial particles using a symmetrical action. This approach knew several applications while following various methods of calculation [17–20].

This paper is organized as follows. In the second section, we start with a brief review of NCQED, from which we derive the noncommutative Dirac equation [21, 22]. Then, in the third section, and due to the smallness of the θ matrix, we formulate the path integral expression relative to NCGF for a Dirac particle in a general external field by adopting the Fradkin–Gitman formalism [15]. As a physical application, in the fourth section, the problem of constant electromagnetic field is considered, in which the perturbative treatment is equivalent to the nonperturbative one due to the linearity of the gauge potential field $A_\mu(x)$. Consequently, the computation leads to an exact path integral expression of NCGF which is in fact similar to the known commutative GF for a Dirac particle moving in a modified constant field written in terms of both constant field $f_{\mu\nu}$ and θ matrix.

2. Noncommutative QED

As has been shown in [21], the construction of the noncommutative Dirac equation coupled to $U(1)$ theory is established with the help of the action that defines the NCQED

$$S = \int \mathrm{d}x \bar{\psi} * (-\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}gA_\mu) - m)\psi - \frac{1}{4}\int \mathrm{d}x F_{\mu\nu}^{(*)} F^{(*)\mu\nu}, \tag{3}$$

where the form of $F_{\mu\nu}^{(*)}$ is

$$F_{\mu\nu}^{(*)} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu]_*. \tag{4}$$

It is clear that the above action is invariant under noncommutative Abelian group $U(1)$ transformations

$$\psi \to U(x) * \psi, \qquad A \to U(x) * A_\mu * U^{-1} + \frac{\mathrm{i}}{g} U(x) * \partial_\mu U^{-1}(x), \tag{5}$$

with

$$U = \exp *(\mathrm{i}\lambda) \qquad \text{and} \qquad U(x) * U^{-1}(x) = 1. \tag{6}$$

The noncommutative Dirac equation that describes the spinorial particle in the presence of an electromagnetic field can be derived from the action (3) and takes the following form:

$$\begin{cases} [\gamma^\mu(-p_\mu + gA_\mu(x)) - m] * \psi(x) = 0, \\ p_\alpha \psi(x) = \mathrm{i}[\partial_\alpha, \psi], \\ [x^\mu, x^\nu] = 0, \end{cases} \tag{7}$$

where γ^μ are the usual Dirac matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ ($\eta^{\mu\nu}$ is the Minkowski metric) and $-g$ is the electron charge.

Now, let us use the definition of the Moyal product (2) up to a first-order θ matrix in equation (7), which leads to

$$[\gamma^\mu \tilde{D}_\mu - m]\psi(x) = 0, \tag{8}$$

where $\tilde{D}_\mu$ is the modified derivative

$$\tilde{D}_\mu = -\left(\delta_\mu^\alpha - \frac{g}{2}\theta^{\beta\alpha}(\partial_\beta A_\mu)\right) p_\alpha + gA_\mu(x). \tag{9}$$

Taking this development to a first-order θ matrix in equation (7) means that either we are treating the noncommutativity as a perturbative effect on the dynamical system, or we are dealing with a linear gauge potential $A_\mu(x)$ as in the case of a constant electromagnetic field.

For a general gauge potential field, we assume that the noncommutative parameter $\theta_{\mu\nu}$ is small ($\theta \precsim (10^{12-13}\ \mathrm{GeV})^{-2}$ [5]) due to the actual constraints on the value of Λ_{NC} which is of the order of TeV, see [6].

3. NC Green's function formulation

The Dirac equation (8) permits us to identify the modified Green's function. However, the NCGF can be represented as a matrix element of the $S^{\mathrm{nc}}(X, P)$ operator

$$S^{\mathrm{nc}}(x, y) = \langle x|\mathbf{S}^{\mathrm{nc}}(X, P)|y\rangle \tag{10}$$

and satisfies the following equation:

$$[\gamma^\mu \tilde{D}_\mu - m]S^{\mathrm{nc}}(x, y) = \delta^4(x - y), \tag{11}$$

where $|x\rangle$ are the eigenvectors of the operator coordinates X^μ, which is the canonical conjugate of $P^\mu = \mathrm{i}\partial^\mu$ verifying $X^\mu|x\rangle = x^\mu|x\rangle$, with

$$\begin{aligned} &P^\mu|p\rangle = p^\mu|p\rangle \qquad \text{and} \qquad \langle x|y\rangle = \delta^4(x - y), \\ &[P^\mu, P^\nu] = [X^\mu, X^\upsilon] = 0, \qquad [X^\mu, P_\upsilon] = \mathrm{i}\delta^\mu_\upsilon. \end{aligned} \tag{12}$$

Thus, equation (11) can be written in operatorial form as

$$(\hat{D}_\mu \gamma^\mu - m)\mathbf{S}^{\mathrm{nc}}(X, P) = I, \tag{13}$$

where the modified Dirac operator is

$$\begin{aligned} \hat{D}_\mu &= -P_\mu + gA_\mu(X) + \frac{g}{2}\theta^{\alpha\beta}(\partial_\alpha A_\mu(X))P_\beta \\ &= -P_\mu + gA_\mu(X) + g(\partial A_\mu) \wedge P \end{aligned} \tag{14}$$

and the $\wedge$ product for any two arbitrary vectors is defined as

$$a \wedge b = \tfrac{1}{2} a_\alpha \theta^{\alpha\beta} b_\beta. \tag{15}$$

The introduction of the definition $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ with $(\gamma^5)^2 = -1$ and the change γ to $\tilde{\gamma}$ as

$$\gamma^\mu \to \tilde{\gamma}^\mu = \gamma^\mu\gamma^5 \tag{16}$$

into equation (13) yield

$$(\hat{D}_\mu \tilde{\gamma}^\mu - m\gamma^5)\tilde{\mathbf{S}}^{\mathrm{nc}}(X, P) = -I, \tag{17}$$

where

$$\tilde{\mathbf{S}}^{\mathrm{nc}}(X, P) = \gamma^5 \mathbf{S}^{\mathrm{nc}}(X, P). \tag{18}$$

The simple solution of equation (17) is given by

$$\tilde{\mathbf{S}}^{\mathrm{nc}}(X,P)=\frac{-I}{\hat{D}_\mu\tilde{\gamma}^\mu-m\gamma^5}=\frac{\hat{D}_\mu\tilde{\gamma}^\mu-m\gamma^5}{(\hat{D}_\mu\tilde{\gamma}^\mu-m\gamma^5)^2}. \tag{19}$$

In order to exhibit the path integral formulation of the operator $\tilde{\mathbf{S}}^{\mathrm{nc}}(X,P)$, we benefit from the Feynman method of proper time which permits us to give the exponential form of equation (19)

$$\tilde{\mathbf{S}}^{\mathrm{nc}}(X,P)=\int_0^\infty \mathrm{d}\lambda\int\mathrm{d}\chi\,\exp\{\mathrm{i}\lambda[(\hat{D}_\mu\tilde{\gamma}^\mu-m\gamma^5)^2+\mathrm{i}\varepsilon]-\mathrm{i}\chi(\hat{D}_\mu\tilde{\gamma}^\mu-m\gamma^5)\}, \tag{20}$$

where λ and χ are, respectively, the even and odd Grassmann proper times.

The operator $\tilde{\mathbf{S}}^{\mathrm{nc}}$ that describes our system becomes

$$\tilde{\mathbf{S}}^{\mathrm{nc}}(X,P)=\int_0^\infty \mathrm{d}\lambda\int\mathrm{d}\chi\,\exp(-\mathrm{i}\tilde{H}), \tag{21}$$

with

$$\tilde{H}=\lambda\big(m^2-\hat{D}^2-\tfrac{1}{2}[\hat{D}_\mu,\hat{D}_\nu]\gamma^\mu\gamma^\nu\big)+(\hat{D}_\mu\tilde{\gamma}^\mu-m\gamma^5)\chi. \tag{22}$$

In a general potential case, we have

$$\begin{aligned}[\hat{D}_\mu,\hat{D}_\nu]&\approx-\mathrm{i}g\tilde{F}_{\mu\nu}+O(\theta^2)\\&=-\mathrm{i}gF_{\mu\nu}(X)-\mathrm{i}g(\partial F_{\mu\nu}(X)\wedge\vec{P})-2\mathrm{i}g^2\partial A_\mu(X)\wedge\partial A_\nu(X)\end{aligned} \tag{23}$$

and

$$\begin{aligned}\hat{D}^2\approx{}&P^2-2gA_\mu P^\mu+g^2A^2-g(\partial A_\mu\wedge P)P^\mu+2g^2A^\mu(\partial A_\mu\wedge P)\\&+g^2(\partial A_\mu\wedge\partial A^\mu)+O(\theta^2),\end{aligned} \tag{24}$$

where we have used the Lorentz gauge

$$[P_\mu,A^\mu]=0. \tag{25}$$

Expression (23) defines the deformed operator related to the electromagnetic strength field which is written in terms of both the usual electromagnetic field $F_{\mu\nu}=\partial_\mu A_\upsilon-\partial_\nu A_\mu$ and the elements of the θ matrix in first order.

Now, we aim to eliminate the operatorial form of equation (21) by subdividing the interval $[x_a,x_b]$ into N parts, then inserting $(N-1)$ projectors

$$\int\mathrm{d}^4x_k|x_k\rangle\langle x_k|=1. \tag{26}$$

Consequently, the matrix element $\langle x_b|\tilde{\mathbf{S}}^c|x_a\rangle$ becomes

$$\begin{aligned}\tilde{S}^{\mathrm{nc}}(x_b,x_a)={}&\lim_{N\to\infty}\int_0^\infty\mathrm{d}\lambda_0\int\mathrm{d}\chi_0\int\mathrm{d}^4x_1\cdots\mathrm{d}^4x_{N-1}\int\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_N\int\mathrm{d}\chi_1\cdots\mathrm{d}\chi_N\\&\times\prod_{k=1}^N\delta(\lambda_k-\lambda_{k-1})\delta(\chi_k-\chi_{k-1})\langle x_k|\,\mathrm{e}^{(-\mathrm{i}H(\lambda_k,\chi_k,x_k)\Delta\tau)}|x_{k-1}\rangle.\end{aligned} \tag{27}$$

Note that

$$x_b=x_N,\qquad x_a=x_0,\qquad \Delta\tau=1/N. \tag{28}$$

The subdivision of the interval $[x_a,x_b]$ allows us to take each exponential term at x_k in its linear form, such that

$$\begin{aligned}\langle x_k|\,\mathrm{e}^{-\mathrm{i}\tilde{H}(\lambda_k,\chi_k,x_k)}|x_{k-1}\rangle&\simeq\langle x_k|1-\mathrm{i}\tilde{H}(\lambda_k,\chi_k,x_k)|x_{k-1}\rangle\\&=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\exp\left\{\mathrm{i}\left[p_k\frac{\Delta x_k}{\Delta\tau}-\mathrm{i}\tilde{H}(\lambda_k,\chi_k,x_k,p_k)\right]\Delta\tau\right\},\end{aligned} \tag{29}$$

where we have inserted the phase space projectors

$$\int \mathrm{d}^4 p_k |p_k\rangle\langle p_k| = 1, \tag{30}$$

with

$$\langle p'|p\rangle = \delta^4(p'-p), \qquad \langle x|p\rangle = \frac{1}{(2\pi)^2}\exp(\mathrm{i}p\cdot x). \tag{31}$$

The scalar product of 4-vectors, denoted by a dot, is $a \cdot b = a_\mu b^\mu$.

By taking into account equation (29), the matrix element $\tilde{S}^{\mathrm{nc}}(x_b, x_a)$ given in expression (27) becomes

$$\begin{aligned}\tilde{S}^{\mathrm{nc}}(x_b,x_a) = \lim_{N\to\infty}\int_0^\infty \mathrm{d}\lambda_0 \int \mathrm{d}\chi_0 \int \mathrm{d}^4x_1\cdots \mathrm{d}^4x_{N-1}\int \frac{\mathrm{d}^4p_1}{(2\pi)^4}\cdots\frac{\mathrm{d}^4p_N}{(2\pi)^4}\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_N \\ \times \int \mathrm{d}\chi_1\cdots\mathrm{d}\chi_N \exp\left\{\mathrm{i}\left[p_k\frac{\Delta x_k}{\Delta\tau} - \mathrm{i}\tilde{H}(\lambda_k,\chi_k,x_k,p_k)\right]\Delta\tau\right\} \\ \times \prod_{k=1}^{N}\delta(\lambda_k-\lambda_{k-1})\delta(\chi_k-\chi_{k-1}).\end{aligned} \tag{32}$$

Let us write the δ function relative to χ and λ variables in their integral forms

$$\begin{aligned}\delta(\chi_k-\chi_{k-1}) &= \mathrm{i}\int \mathrm{d}p_{\chi_k}\,\mathrm{e}^{\{\mathrm{i}p_{\chi_k}(\chi_k-\chi_{k-1})\}} \\ \delta(\lambda_k-\lambda_{k-1}) &= \mathrm{i}\int \mathrm{d}p_{\lambda_k}\,\mathrm{e}^{\{\mathrm{i}p_{\lambda_k}(\lambda_k-\lambda_{k-1})\}},\end{aligned} \tag{33}$$

where p_{χ_k} and p_{λ_k} are, respectively, the odd and even Grassmann variables. By introducing these forms into (32), we get

$$\begin{aligned}\tilde{S}^{c}(x_b,x_a) = T\int_0^\infty \mathrm{d}\lambda_0\int \mathrm{d}\chi_0\int_{x_a}^{x_b}\mathrm{D}x\int \mathrm{D}p\mathrm{D}\lambda\mathrm{D}p_\lambda\mathrm{D}p_\chi\mathrm{D}\chi\exp\left\{\mathrm{i}\int_0^1\left[\lambda\left(\tilde{D}^2(x,p)\right.\right.\right. \\ \left. -m^2 - \mathrm{i}\frac{g}{2}\tilde{F}_{\mu\nu}(x,p)\gamma^\mu\gamma^\nu + (-\hat{D}_\mu(x,p)\tilde{\gamma}^\mu + m\gamma^5)\chi\right. \\ \left.\left. + p\cdot\dot{x} + p_\lambda\dot{\lambda} + p_\chi\dot{\chi}\right]\mathrm{d}\tau\right\}.\end{aligned} \tag{34}$$

Here T is the known chronological operator related to the time dependence of $\tilde{\gamma}$ matrices, and in order to cancel it, we use these formulae [15]

$$T\exp\{f(\tilde{\gamma}^n(\tau))\} = \exp\left\{f\left(\frac{\partial_l}{\partial\rho_n}\right)\right\}T\exp\left\{\int_0^1\rho_n\tilde{\gamma}^n\,\mathrm{d}\tau\right\}\bigg|_{\rho=0}, \tag{35}$$

with

$$\begin{aligned}T\exp\left\{\int_0^1\rho_n\tilde{\gamma}^n\,\mathrm{d}\tau\right\} = \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\partial_l}{\partial\zeta^n}\right) \\ \times\int_E\tilde{D}\psi\exp\left\{\int_0^1(\psi_n\dot{\psi}^n - 2\mathrm{i}\rho_n\psi^n)\,\mathrm{d}\tau + \psi_n(1)\psi^n(0)\right\}\bigg|_{\zeta=0}.\end{aligned} \tag{36}$$

$\rho_n(\tau)$ are called the currents and $\psi(\tau)$ are the Grassmann variables with the measure $\tilde{D}\psi$

$$\tilde{D}\psi = D\psi\left[\int_E D\psi\exp\left\{\int_0^1\psi_n\dot{\psi}^n\,\mathrm{d}\tau\right\}\right]^{-1}, \tag{37}$$

thus, ζ is an odd Grassmann variable, with the integration over ψ verifying the boundary condition

$$E = \psi^n(0) + \psi^n(1) = \zeta^n, \qquad n = \overline{0,3} \tag{38}$$

Finally, the expression of NCGF relative to the relativistic spinorial particle under the effect of an electromagnetic field is

$$\begin{aligned}\tilde{S}^{\mathrm{nc}}(x_b, x_a) = \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\partial_l}{\partial \zeta^n}\right) \int_0^\infty \mathrm{d}\lambda_0 \int \mathrm{d}\chi_0 \int_E \tilde{D}\psi \int_{x_a}^{x_b} \mathrm{D}x \int \mathrm{D}p\, \mathrm{D}\lambda\, \mathrm{D}p_\lambda\, \mathrm{D}p_\chi\, \mathrm{D}\chi \\ \times \exp\left\{\mathrm{i}\int_0^1 [\lambda(\tilde{D}^2 - m^2 + 2\mathrm{i}g\tilde{F}_{\mu\nu}(x,p)\psi^\mu\psi^\nu) \right. \\ + 2\mathrm{i}(\hat{D}_\mu(x,p)\psi^\mu - m\psi^5)\chi + p\cdot\dot{x} + p_\lambda\dot{\lambda} + p_\chi\dot{\chi} \\ \left. - \mathrm{i}\psi_n\dot{\psi}^n]\,\mathrm{d}\tau + \psi_n(1)\psi^n(0)\right\}\Big|_{\zeta=0}.\end{aligned} \tag{39}$$

Expression (39) contains the entire dynamics of the system in a noncommutative space framework, and obviously, its computation is equivalent to the resolution of the modified Dirac equation (8).

Similarly to the usual formulation worked out in commutative space [15, 16], we remark that the action of expression (39) is made up of two parts, fermionic (or called spin factor) and bosonic. The fermionic part describes the spin interaction with the exterior field. The bosonic part (which does not contain the Grassmann variables) is called the dynamical part which describes the matter interaction of the particle without the contribution of spin.

4. The constant electromagnetic field

By using the formulation shown in the previous section, we compute the exact NCGF for a spinorial particle interacting with an external constant electromagnetic field. The counterpart of this problem given in commutative space has already been treated by Gitman [23] and Cruz [24] using the Fradkin–Gitman formalism. We will show that their result will be a shortcut towards a total deduction of the influence of noncommutativity on the dynamics of a particle in interaction with a constant electromagnetic field.

Initially, let us give the gauge potential $A_\mu(x)$ that defines the constant field $f_{\mu\nu}$ as follows:

$$A_\mu(x) = -\tfrac{1}{2} f_{\mu\nu} x^\nu, \tag{40}$$

in this case, the problem of many derivatives does not appear since we have

$$\partial_\alpha \partial_\beta A_\mu(x) = 0, \tag{41}$$

which naturally leads to a first θ order expanding series in equation (7), from which the modified Dirac operator (14) acquires an exact form

$$\hat{D}_\mu = -\tilde{P}_\mu + gA_\mu(X), \tag{42}$$

with

$$\tilde{P}_\mu = M_\mu^\alpha P_\alpha, \qquad M_\mu^\alpha = \left(\delta_\mu^\alpha + \frac{g}{4} f_{\mu\rho}\theta^{\rho\alpha}\right). \tag{43}$$

Hence, the modified antisymmetric field is

$$[\hat{D}_\mu, \hat{D}_\upsilon] = -\mathrm{i}g\tilde{f}_{\mu\nu} = -\mathrm{i}g\left(f_{\mu\nu} - \frac{g}{2}(f\wedge f)_{\mu\nu}\right). \tag{44}$$

Note that $\wedge$ product of two matrices is given by

$$(f \wedge f)_{\mu\nu} = \tfrac{1}{2} f_{\mu\alpha}\theta^{\alpha\beta} f_{\beta\nu}. \tag{45}$$

Since the generalized Lorentz gauge remains verified

$$[\tilde{P}_{\mu}, A^{\mu}] = 0, \tag{46}$$

we get

$$\hat{D}^2 = \tilde{P}^2 - 2gA\cdot\tilde{P} + g^2A^2. \tag{47}$$

Let us substitute the developed equations (42), (44) and (47) into equation (22), then eliminate the obtained operatorial form using the projectors (26) and (30). By following the same remaining steps given in the third section, we get the exact path integral expression of NCGF relative to a Dirac particle in a constant electromagnetic field

$$\begin{aligned}\tilde{S}^{\rm nc}(x_b, x_a) = {}& \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\partial_l}{\partial\theta^n}\right)\int_0^{\infty} \mathrm{d}\lambda_0 \int \mathrm{d}\chi_0 \int_{x_a}^{x_b} \mathrm{D}x \int \mathrm{D}p\,\mathrm{D}\lambda\,\mathrm{D}p_\lambda\,\mathrm{D}p_\chi\,\mathrm{D}\chi \\ & \times \exp\left\{\mathrm{i}\int_0^1 [\lambda(\tilde{p}^2 - 2gA\cdot\tilde{p} + g^2A^2 - m^2 + 2\mathrm{i}g\tilde{f}_{\mu\nu}\psi^\mu\psi^\nu) \right.\\ & + 2\mathrm{i}((-\tilde{p}+gA)_\mu\psi^\mu - m\psi^5)\chi + p\cdot\dot{x} + p_\lambda\dot{\lambda} + p_\chi\dot{\chi} - \mathrm{i}\psi_n\dot{\psi}^n]\,\mathrm{d}\tau \\ & \left. + \psi_n(1)\psi^n(0)\right\}\Big|_{\zeta=0}.\end{aligned} \tag{48}$$

We can express the scalar product $(\dot{x}\cdot p)$ in terms of $\tilde{p}$ instead of impulsion p as

$$\dot{x}\cdot p = \dot{x}_\nu\eta^\nu_\mu p^\mu = \dot{x}_\nu(M^{-1}\cdot M)^\nu_\mu p^\mu = \dot{\widehat{x}}\cdot\tilde{p}, \tag{49}$$

with

$$\widehat{x}_\mu = x_\nu(M^{-1})^\nu_\mu \qquad \text{and} \qquad \tilde{p}^\nu = M^\nu_\mu p^\mu. \tag{50}$$

After performing the following changes:

$$\lambda \to \frac{e}{2}, \qquad p_\lambda \to 2p_e \qquad \text{and} \qquad \tilde{p} \to -\tilde{p} - gA(x) - \frac{\dot{\widehat{x}}}{e} \tag{51}$$

into expression (48), and integrating successively over (p_e, e), then over (p_χ, χ), we get

$$\begin{aligned}\tilde{S}^{\rm nc}(x_b, x_a) = {}& \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\partial_l}{\partial\theta^n}\right)\int_0^{\infty} \mathrm{d}c_0 \int \mathrm{d}\chi_0 M(e_0)\int_{x_a}^{x_b} \mathrm{D}x\,\tilde{D}\psi \\ & \times \exp\left\{\mathrm{i}\int_0^1 \left[-\frac{\dot{\widehat{x}}^2}{2e_0} - \frac{e_0}{2}m^2 - g\dot{\widehat{x}}\cdot A(x) + \mathrm{i}e_0 g\tilde{f}_{\mu\nu}\psi^\mu\psi^\nu\right.\right. \\ & \left.\left. + \mathrm{i}\left(\frac{\dot{\widehat{x}}.\psi}{e_0} - m\psi^5\right)\chi_0 - \mathrm{i}\psi_n\dot{\psi}^n\right]\mathrm{d}\tau + \psi_n(1)\psi^n(0)\right\}\Bigg|_{\zeta=0}.\end{aligned} \tag{52}$$

The measure $M(e_0)$ is

$$M(e_0) = \int \mathrm{D}p \exp\left\{\frac{\mathrm{i}}{2}\int_0^1 e_0\tilde{p}^2\,\mathrm{d}\tau\right\}, \tag{53}$$

The term $\dot{\widehat{x}}\cdot A(x)$ appearing in the action of expression (52) is inhomogeneous because of its dependence on x. In order to obtain an action which is only expressed in terms of $\widehat{x}$ and $\tilde{p}$ instead of the usual position and impulsion vectors x and p, we make

$$\dot{\widehat{x}}^\mu f_{\mu\nu}x^\nu = \dot{\widehat{x}}^\mu f_{\mu\nu}\eta^\nu_\alpha x^\alpha = \dot{\widehat{x}}^\mu f_{\mu\nu}M^\nu_\beta(M^{-1})^\beta_\alpha x^\alpha = \dot{\widehat{x}}^\mu\tilde{f}_{\mu\beta}\widehat{x}^\beta. \tag{54}$$

The performed transformations from (x, p) to $(\hat{x}, \tilde{p})$ lead to the unchanged integrations

$$\int \mathrm{D}p\int_{x_a}^{x_b}\mathrm{D}x = \int \mathrm{D}\tilde{p}\int_{\widehat{x}_a}^{\widehat{x}_b}\mathrm{D}\widehat{x}, \tag{55}$$

where the boundary conditions (x_a, x_b) undergo a translation to other positions $(\widehat{x}_a, \widehat{x}_b)$, such as

$$\widehat{x}_\mu^{(a,b)} = x_\alpha^{(a,b)} (M^{-1})_\mu^\alpha. \tag{56}$$

By considering equations (54) and (55), expression (52) can be written as

$$\begin{aligned}\tilde{S}^{\mathrm{nc}}(x_b, x_a) = \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\partial_l}{\partial\theta^n}\right) \int_0^\infty \mathrm{d}e_0 \int \mathrm{d}\chi_0 \tilde{M}(e_0) \int_{\widehat{x}_a}^{\widehat{x}_b} \mathrm{D}\widehat{x}\tilde{D}\psi \\ \times \exp\left\{\mathrm{i}\int_0^1 \left[-\frac{\dot{\widehat{x}}^2}{2e_0} - \frac{e_0}{2}m^2 - g\widehat{x}^\mu \tilde{f}_{\mu\nu}\widehat{x}^\nu + \mathrm{i}e_0 g \tilde{f}_{\mu\nu}\psi^\mu\psi^\nu \right.\right. \\ \left.\left. + \mathrm{i}\left(\frac{\dot{\widehat{x}}.\psi}{e_0} - m\psi^5\right)\chi_0 - \mathrm{i}\psi_n\dot{\psi}^n\right] \mathrm{d}\tau + \psi_n(1)\psi^n(0)\right\}\bigg|_{\theta=0},\end{aligned} \tag{57}$$

with the measure $\tilde{M}(e)$ is

$$\tilde{M}(e) = \int \mathrm{D}\tilde{p} \exp\left\{\frac{\mathrm{i}}{2}\int_0^1 e_0\tilde{p}^2 \,\mathrm{d}\tau\right\}. \tag{58}$$

Finally, expression (57) is the known Green function for a Dirac particle interacting with a constant electromagnetic field $\tilde{f}_{\mu\nu}$ in commutative space. Its result is already computed in [23, 24]; then, the result is

$$\begin{aligned}\tilde{S}^{\mathrm{nc}}(x_b, x_a) = \frac{1}{32\pi^2}\int_0^\infty \mathrm{d}e_0 \left(\det \frac{\sinh \frac{ge_0\tilde{f}}{2}}{g\tilde{f}}\right)\left(\det\cosh\frac{ge_0\tilde{f}}{2}\right)^{1/2} \\ \times \exp\left\{\frac{\mathrm{i}g}{2}\widehat{x}_b\cdot\tilde{f}\widehat{x}_a - \frac{\mathrm{i}}{e_0}m^2 - \frac{\mathrm{i}g}{4}(\widehat{x}_b - \widehat{x}_a)\cdot\tilde{f}\coth\left(\frac{ge_0\tilde{f}}{2}\right)(\widehat{x}_b - \widehat{x}_a)\right\} \\ \times\left\{m + \frac{g}{2}(\widehat{x}_b - \widehat{x}_a)\cdot\tilde{f}\left(\coth\frac{ge_0\tilde{f}}{2} - 1\right)\gamma\right\} \\ \times\left\{1 - \frac{\mathrm{i}}{2}\left(\tanh\frac{ge_0\tilde{f}}{2}\right)_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{4}\left(\tanh\frac{ge_0\tilde{f}}{2}\right)^*_{\mu\nu}\left(\tanh\frac{ge_0\tilde{f}}{2}\right)^{\mu\nu}\gamma^5\right\}.\end{aligned} \tag{59}$$

However, the calculation of NCGF for a Dirac particle in a constant field $f_{\mu\nu}$ is reduced to the calculation of the usual Green function for a Dirac particle in a modified constant field $\tilde{f}_{\mu\nu}$ treated in commutative space.

As has been shown in the study of the pair production rate induced by an external electric field [21], the threshold electric field is decreased by the noncommutativity effects. In our case, the result (58) shows us that the influence of noncommutativity on the system of a Dirac particle moving in a constant electromagnetic field $f_{\mu\nu}$ has the same effect as an additional constant field proportional to $(f \wedge f)_{\mu\nu}$ (deduced from equation (44)) on this system worked out in commutative space. Therefore, the particle seems to be in interaction with a modified constant field $\tilde{F}_{\mu\nu}$, i.e.

$$\tilde{S}^{\mathrm{nc}}(x_b, x_a) \equiv \tilde{S}^{\mathrm{nc}}(x_b, x_a, f) = S^c(\widehat{x}_a, \widehat{x}_b, \tilde{f}), \tag{60}$$

where $S^c(\widehat{x}_a, \widehat{x}_b, \tilde{f})$ denotes the commutative Green function that connects the initial state at $\widehat{x}_a$ and the final one at $\widehat{x}_b$, relative to a Dirac particle in constant field $\tilde{f}_{\mu\nu}$.

5. Conclusion

This work is devoted essentially to the study of the influence of space noncommutativity on the dynamics of a relativistic particle interacting with an external constant field $f_{\mu\nu}$, in the context of the path integral approach. In order to derive the modified Dirac equation that describes the relativistic quantum mechanics in a noncommutative space framework, we started with a brief introduction of the noncommutative Abelian theory relative to NCQED. The new Dirac equation is expressed in terms of expanding series in θ matrix due to the star product definition, and in which the problem of many derivatives could occur in some case of electromagnetic field (Coulomb potential, etc). For this reason, and since the effect of the noncommutativity is very small, we have formulated the path integral expression relative to NCGF for a Dirac particle in a general external field while neglecting θ at second order. The exact resolution of the noncommutative Dirac equation, without taking the noncommutativity as a perturbative effect, is possible for quadratic electromagnetic fields (constant field, harmonic oscillator potential, etc) (see [7]). However, for the constant electromagnetic field $f_{\mu\nu}$ case, the expression of NCGF has an exact path integral formulation, and it is shown that its expression is similar to that of commutative GF for a Dirac particle moving in a modified constant field $\tilde{f}_{\mu\nu}$ depending on both θ matrix and $f_{\mu\nu}$. Hence, the effect of noncommutativity is manifested as an additional constant field on the particle. This result justifies the decrease in the threshold electric field because of the noncommutativity reported in [21].

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